1. Let $N(p): S \to S^2$ be gauss map of S

We claim $dN_p = 0 \quad \forall p \in S$, then $N(p)$ is constant $\forall p \in S$ and so S is contained in a plane

Let $p_0 \in S$ and $\{e_1, e_2\}$ be orthonormal basis of $T_{p_0}S$

Then
$$0 = H = h_{11} + h_{22} = A(e_1,e_1) + A(e_2,e_2) \qquad h = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$$

$$0 = K = h_{11}h_{22} - h_{12}h_{21} \qquad h_{ij} = A(e_i, e_j)$$
$$= A(e_1,e_1)A(e_2,e_2) - A(e_1,e_2)A(e_2,e_1)$$
$$= A(e_1,e_1)A(e_2,e_2) - A(e_1,e_2)^2 \qquad (\because A(e_i,e_j) = A(e_j,e_i))$$

$$0 \leq (A(e_1,e_1) - A(e_2,e_2))^2 = (A(e_1,e_1) + A(e_2,e_2))^2 - 4A(e_1,e_1)A(e_2,e_2)$$
$$= -4A(e_1,e_1)A(e_2,e_2)$$
$$= -4A(e_1,e_2)^2 \leq 0$$

Hence $A(e_1,e_1) = A(e_2,e_2) = A(e_1,e_2) = 0$

$\langle -dN_{p_0}(e_1), e_1 \rangle = A(e_1,e_1) = 0$, $\langle -dN_{p_0}(e_1), e_2 \rangle = A(e_1,e_2) = 0$

$\Rightarrow \quad dN_{p_0}(e_1) = 0$

Similarly, $dN_{p_0}(e_2) = 0$

So $dN_{p_0} = 0$

2. Let $X(u,v) = (a\sin v\cos u, b\sin v\sin u, c\cos v) \qquad u \in (0,2\pi), v \in (0,\pi)$

$$X_u = (-a\sin v\sin u, b\sin v\cos u, 0)$$
$$X_v = (a\cos v\cos u, b\cos v\sin u, -c\sin v)$$
$$X_{uu} = (-a\sin v\cos u, -b\sin v\sin u, 0)$$
$$X_{vv} = (-a\sin v\cos u, -b\sin v\sin u, -c\cos v)$$
$$X_{uv} = (-a\cos v\sin u, b\cos v\cos u, 0)$$
$$N = \frac{X_u \times X_v}{|X_u \times X_v|} = \frac{1}{|X_u \times X_v|}(-bc\sin^2 v\cos u, -ac\sin^2 v\sin u, -ab\cos v\sin v)$$

$$K = \frac{\det(h)}{\det(g)} \quad \text{where } h = \begin{pmatrix} \langle N, X_{uu} \rangle & \langle N, X_{uv} \rangle \\ \langle N, X_{vu} \rangle & \langle N, X_{vv} \rangle \end{pmatrix} \qquad g = \begin{pmatrix} \langle X_u, X_u \rangle & \langle X_u, X_v \rangle \\ \langle X_v, X_u \rangle & \langle X_v, X_v \rangle \end{pmatrix}$$

$K>0$ iff $\det(h)>0$ as $\det(g)>0$

$$h=\begin{pmatrix}\langle N,X_{uu}\rangle & \langle N,X_{uv}\rangle \\ \langle N,X_{uv}\rangle & \langle N,X_{vv}\rangle\end{pmatrix} = \frac{1}{|X_u\times X_v|}\begin{pmatrix}\langle X_u\times X_v, X_{uu}\rangle & \langle X_u\times X_v, X_{uv}\rangle \\ \langle X_u\times X_v, X_{uv}\rangle & \langle X_u\times X_v, X_{vv}\rangle\end{pmatrix}$$

$$\det(h) = \frac{1}{|X_u\times X_v|^2}\left|\cdots\cdots\right|$$

$\langle X_u\times X_v, X_{uu}\rangle = abc\sin^3 v$, $\langle X_u\times X_v, X_{uv}\rangle = 0$, $\langle X_u\times X_v, X_{vv}\rangle = abc\sin v$

$\Rightarrow \det(h) = \frac{1}{|X_u\times X_v|^2}(abc)^2\sin^4 v>0$

Since $X(u,v)$ does not cover the whole S, we may need another chart such as $Y(u,v)=(-a\sin v\cos u, b\cos v, c\sin v\sin u)$ and repeat the same procedure to show S has positive Gauss curvature at every point

3. We assume α is p.b.a.l.

Let $p=\alpha(t_0)$, $v=\alpha'(t_0)$

Let $w\in T_pS$ such that $\{v,w\}$ is orthonormal basis for T_pS

$$K_p = \det\begin{pmatrix}A(v,v) & A(v,w)\\ A(w,v) & A(w,w)\end{pmatrix}$$

$dN_p(v) = \frac{d}{dt} N\circ\alpha(t)\big|_{t=t_0} = 0$ as $\alpha(I)\subset P$, N is constant on α

So $A(v,v)=0$, $A(v,w)=0$, $A(w,v)=0$

$K_p = A(v,v)A(w,w)-A(v,w)A(w,v)=0$

4. Let $\alpha: I\to S$ be the straight line p.b.a.l.

Let $p\in l$, $\alpha(t_0)=p$, $\alpha'(t_0)=v$ and $w\in T_pS$ s.t. $\{v,w\}$ is orthonormal basis for T_pS

$K_p = A(v,v)A(w,w)-A(v,w)^2$

We claim $A(v,v)=0$, then $K_p = 0-A(v,w)^2\le 0$

$A(v,v) = \langle -dN_p(v), v\rangle$

$= \langle -\frac{d}{dt} N\circ\alpha(t)\big|_{t=t_0}, \alpha'(t)\big|_{t=t_0}\rangle$

Since $\langle N\circ\alpha(t), \alpha'(t)\rangle = 0 \quad \forall\ t \in I$

$$\langle \tfrac{d}{dt} N\circ\alpha(t), \alpha'(t)\rangle + \langle N\circ\alpha(t), \alpha''(t)\rangle = 0$$

$$\langle \tfrac{d}{dt} N\circ\alpha(t)|_{t=t_0}, \alpha'(t)|_{t=t_0}\rangle = -\langle N\circ\alpha(t)|_{t=t_0}, \alpha''(t)|_{t=t_0}\rangle = 0$$

(as $\alpha''(t) = 0$)

So $A(v,v) = 0$